

Invariant quantization in warped spacetimes

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Abstract

We argue that quantum theory in curved spacetime should be invariant under the continuous spacetime symmetries that are connected with the identity. For the most typical warped-product spacetimes, we prove that such invariance can be actually implemented, at least at the level of first quantization. Indeed, in the framework of warped spacetimes, we can ensure the isometric invariance of the projection operator that selects the positive-energy piece of any solution to the wave equation. Of course, for a given spacetime, this operator is not unique, but the requirement of invariance under spacetime isometries drastically reduces the arbitrariness. Quantum theory of free fields additionally requires that this linear operator admits a kernel. We briefly discuss conditions implying that an invariant kernel actually exists, and we characterize the general form it can take on.

1 Introduction

There are two ways for constructing a quantum theory of free fields.

The traditional formulation starts with a single-particle Hilbert space, then considers Fock space, and further introduces field operators through creation and annihilation operators associated with the so-called "two-point function".

In contrast, the algebraic formulation, immediately starts from a C^* -algebra of fundamental observables; then, it recovers the concept of state vector with help of sophisticated mathematical developments.

From a very general point of view, it is true that the algebraic approach is more powerful: in many cases, the various possible quantizations permitted by the traditional formulation are not all unitarily equivalent among themselves, whereas the inequivalent representations correspond to C^* -algebras of fundamental observables that are, however, isomorphic.

Nevertheless, in this article, we shall stick to the traditional approach, first because of a personal preference for the most intuitive setting, and also because we keep in

mind that, for a large class of spacetimes (for instance, in the case of a spatially closed universe, see [1] pp. 96-97) the traditional formulation of quantum field theory remains essentially as powerful as the algebraic approach.

We naturally start with "first quantization", and consider the wave equation for a c -number wave function. Indeed, in spite of well-known limitations concerning the physical interpretation of one-body relativistic quantum mechanics, it is of conceptual interest to investigate how far one can go without inconsistencies by developing a single-particle quantum theory for its own right. And even from the viewpoint of quantum field theory, a careful construction of the one-particle sector is a technical prerequisite.

In fact, in the traditional approach, it is easy to disentangle first quantization from the issues that are genuinely concerned with quantum field theory; this way of proceeding allows to avoid unnecessary complications. This remark typically applies to the problem of separating positive from negative-energy solutions. So the first step consists in setting up the Hilbert space for a single particle; the construction of Fock space will be performed ultimately, with help of a positive-energy kernel associated with the projection operator.

Insofar as *scalar* particles are concerned, one starts from the Klein-Gordon (KG) equation. Its solutions form a linear space \mathcal{K} endowed with a natural sesquilinear form; this form is by no means positive. But a positive definite scalar product (and ultimately a Hilbert space) will be obtained further, provided one is able to perform a convenient splitting of \mathcal{K} into positive and negative-frequency solutions.

This splitting corresponds to the action of a projection linear operator Π^+ , or equivalently by that of a so-called *complex-structure operator* J which must enjoy a special algebraic property with respect to the sesquilinear form (in fact J must be, in certain sense, *positive* with respect to the underlying skew-symmetric form w).

In as much as linear operators acting in \mathcal{K} can be represented by bi-scalar kernels, an alternative formulation of the splitting can also be made in terms of a positive-frequency kernel D^+ which is sometimes referred to as the two-point Wightman "function". In contradistinction to other propagators and Green functions, in general this kernel *is not* uniquely provided by the geometry. Since all the properties of the quantum theory of free fields are encoded in it, the knowledge of this kernel is usually considered as providing a definition of the vacuum.

For an arbitrary given spacetime, the existence of a splitting which separates positive-frequency solutions from negative-frequency solutions is not problematic, but its lack of unicity has motivated an abundant literature. First results were obtained about static and stationary spacetimes [2] [3]; apart from these special cases, there is generally no preferred splitting. Soon it was realized that, under very general assumptions including global hyperbolicity, each spacelike Cauchy surface allows for a different definition of the frequency splitting [4].

This result was cast into a rigorous form by C. Moreno [5] who proved the symplectic (resp. unitary) equivalence of these different definitions; he took a step further by directly constructing the complex structures in terms of their kernels, according to the Lichnerowicz program [6].

Thus, insofar as one ignores the issue of symmetries, the construction of quantum mechanics in a general spacetime is, to a very large extent, a solved problem.

But, in the framework of General Relativity, it is natural to demand that quantum theory be invariant under spacetime isometries. We have elsewhere [7] proposed a principle of isometric invariance:

Quantum mechanics of free particles must be invariant under all spacetime isometries continuously connected with the Identity

This condition is a natural generalization of the requirement of Poincaré invariance which is at the very foundation of quantum field theory in Minkowski spacetime. Here, spacetime symmetries generate first integrals for the motion of a test particle (this is true at least for minimal coupling). Insofar as these constants of the motion have a physical meaning, they should be represented by operators acting in the positive-frequency sector (in the hope of ultimately being implemented as essentially self-adjoint operators). They must therefore map this sector into itself, which requires that they commute with Π^+ . Thus, in a spacetime admitting Killing vectors, imposing isometric invariance does not remove but substantially restricts the arbitrariness in the choice of Π^+ . For a given spacetime of this kind, the first question, of course, is whether an invariant quantization is possible at all. At first sight, it may be possible to construct an invariant frequency splitting by using an invariant Cauchy surface. This procedure is obviously limited to cases where all Killing vectors are spacelike, and faces the question as to determine a Cauchy surface which is invariant under the action of all of them.

Note that the invariance of Π^+ may be ensured by that of its kernel D^+ . But, whereas the isometric invariance of retarded and advanced Green functions has been known for a long time [8], most available results ensuring the existence of D^+ in a large class of spacetimes tell practically nothing about the possible invariance of this kernel under isometries. There are a few exceptions: the issue of constructing an invariant vacuum has been discussed for de Sitter spacetime [9]; see a remark in Birell and Davies [10] about invariance of the Wightman "function" in an asymptotically static, spatially flat FRW universe. In a previous work [7] we presented the case of generalized FRW universes.

In this article, we focus on warped product manifolds because they have the following properties:

First, their isometry group is, to a large extent, under control, owing to the works of Carot and da Costa [11] and M. Sánchez [12].

Second, the wave equation in such spacetimes admits a remarkable constant of the motion, which allows for considering sectors of *mode solutions*. This situation leads to constructing an invariant energy-splitting separately for each mode. In fact, the wave equation undergoes a separation of the variables and one is left with a reduced equation involving a minor number of degrees of freedom.

This paper is organized as follows: Section 2 contains basic formulas, a display of the notation and terminology. Warped spacetimes and known results about their isometries are recalled in Section 3. In Section 4 we perform the separation of the variables according to the notion of modes and reformulate our primitive problem

in a reduced manner, taking advantage of the fact that each mode is a tensor-product. Section 5 is devoted to general facts of bilinear algebra about skew forms, sesquilinear forms and complex structures in a tensor product. In Section 6 we characterize a "canonical" form of the splitting we were looking for, and perform a sum over modes in order to construct the sector of positive-frequency solutions. Most part of this article is devoted to the single-particle problem. We are thus concerned with the KG equation for a c-number field. Next step would be the construction (in invariant manner) of creation and annihilation operators in Fock space. This program requires that our splitting actually admits a kernel, and that this kernel in turn respects isometric invariance. Kernels and two-point functions are briefly discussed in Section 7. The final Section is about remarks and conclusions. Appendix 1 deals with the KG equation with a source term, Appendix 2 offers a summary of useful elementary calculations.

2 Basic formulas

Spacetime is considered as given, and we are not concerned with stress-energy tensors. Throughout this paper we consider smooth metrics, smooth functions, and linear operators acting in various functional spaces. Our point of view is that of differential geometry; operators and eigenvalues are understood in the sense of the geometric spectral theory [13].

As long as possible, we remain within the framework of multilinear algebra and postpone several issues involving the continuity of the operators. But, of course, continuity becomes essential as soon as one wishes to associate Π^+ with a kernel $D^+(x, y)$ (a distribution), as is necessary in order to consider Fock space.

The main tool for connecting with topologic matters is the observation that, the operators we construct with help of Cauchy surfaces are actually continuous because the functions determined by their Cauchy data depend continuously on them.

We consider here the Klein-Gordon equation

$$(\nabla^2 + m^2)\psi = 0 \quad (1)$$

for a wave function $\psi(x)$ describing the minimal coupling of a scalar particle with gravity. Complex (resp. real) solutions to (1) live in an infinite dimensional vector space \mathcal{K} (resp. \mathcal{K}^R). The real and skew-symmetric form

$$\varpi(\phi, \psi) = \int_{\Sigma} (\phi \nabla^{\mu} \psi - \psi \nabla^{\mu} \phi) d\Sigma_{\mu} \quad (2)$$

is conservative with respect to changes of hypersurface Σ , provided ϕ and ψ are solutions to the KG equation. Under very general assumptions, ϖ is in fact a *symplectic form*.

For complex ϕ and ψ , the sesquilinear form

$$(\phi; \psi) = -i\varpi(\phi^*, \psi) = \int j^{\mu}(\phi, \psi) d\Sigma_{\mu} \quad (3)$$

is *not* positive definite. Thus, in order to exhibit a candidate for one-particle pre-Hilbert space, the linear space of solutions \mathcal{K} must be split as the direct sum of two subspaces. In one of them (further completed and identified as the positive-frequency sector) the restriction of $(\phi; \psi)$ must be definite positive. The trouble is that such splitting is not unique and, except for the case of stationary spacetimes [2] [3] there is no natural way to select a preferred choice. Nevertheless, criteria for the determination of this splitting have been given soon, either in terms of finding a real linear operator J with $J^2 = -1$, determining a complex structure in the space of real solutions [14], or equivalently in terms of a projector $\Pi^+ = \frac{1}{2}(1 + iJ)$ which projects any complex solution into the positive-frequency subspace.

In fact, J must be *positive* with respect to the skew-symmetric form ϖ . This property is more restrictive than simply leaving the skew form invariant; it means both following conditions

$$\varpi(J\phi, J\psi) = \varpi(\phi, \psi), \quad \varpi(\phi, J\phi) > 0 \quad (4)$$

whenever ϕ is a real solution. The latter condition (4) has an obvious connexion with the need to make the sesquilinear form positive on some subspace. The former ensures that Π^+ is a symmetric operator with respect to the sesquilinear form (3), which in turn entails that the positive and negative-frequency subspaces are mutually orthogonal.

The result obtained by Ashtekar and Magnon [4] for minimal coupling consists in the construction of an admissible J_Σ for every Cauchy surface Σ . This was possible because, assuming that spacetime is globally hyperbolic, each solution to the KG equation can be uniquely determined by its value and that of its normal derivative on Σ . In fact, owing to the global generalization [15] of Dronne's theorem, this property of the KG equation holds true for a wide class of hyperbolic second order partial differential equations; in particular, it is true also for the KG equation with a source term (see Appendix 1).

Given one isometric transformation T of spacetime, we are led to investigate under which conditions J_Σ is actually invariant by T . Naturally, we find that this is the case when T maps the Cauchy surface Σ into itself (Appendix). For some particular spacetimes, one can find a Cauchy surface invariant by *the whole group* of isometries. But we cannot always expect this situation. For instance, this is impossible as soon as (V, g) admits a timelike Killing vector. In that case however, it is possible to define directly a splitting of the wave functions according to the sign of the energy; the projection operator associated with this splitting obviously corresponds to an appropriate complex-structure operator J .

2.1 The positive-frequency kernel

Insofar as spacetime is globally hyperbolic, the retarded and advanced Green functions and (by difference) the Jordan-Pauli commutator "function" are unambiguously determined by the geometry.

More problematic is the kernel D^\pm which (if it exists) defines the positive-frequency

(resp. negative-frequency) solutions to the KG equation, through the formula

$$(\Psi^\pm)(y) = (D^\pm(y, x); \Psi(x)) = \int j^\alpha(D^\pm(y, x)\Psi(x)) d\Sigma_\alpha \quad (5)$$

where $\Psi^\pm = \Pi^\pm \Psi$ is the positive-frequency part of Ψ , and y is an arbitrary point of V . We adopt a generalized Eistein notation: in expressions of the form $(.;.)$ we make the convention that integration must be performed over the variable which is twice repeated.

It is clear that D^\pm is a kernel for Π^\pm . Note that D^\pm is a "two-point function" (actually, a distribution) and must satisfy the KG equation in both arguments. Our interest for this kernel is motivated by the fact that the field operator must be defined through creation and annihilation operators associated with the one-particle state D^+ .

Naturally, when D^+ exists, we can always recover the projector Π^+ , for Ψ^+ is given by (5). But the converse requires some care: given a complex-structure operator, or equivalently the projector Π^+ , it is tempting to formally write (5) and claim that it defines D^+ as a distribution; indeed associating Ψ^+ to Ψ obviously defines a linear functional. But in order to make up a distribution, this functional should additionally be in some sense *continuous*, which requires some topology on \mathcal{K} . However, as much as possible, we postpone topological considerations to the end of this study and simply use the theory of linear operators in a vector space.

Most results available in the litterature are formulated purely in terms of J or Π^+ , and disregard the kernel. In contradistinction, C. Moreno has exhibited complex structures by directly constructing their kernel.

3 Warped spacetimes

A warped spacetime is a product manifold $V = V_1 \times V_2$, endowed with a metric (omitting indices)

$$g = \alpha \oplus (-S)\gamma \quad (6)$$

where α and γ , respectively, are metric tensors on V_1 and V_2 , and S is a positive function on V_1 . Usually \sqrt{S} is referred to as the warping factor, and one sets $S = \exp(2\Theta)$.

In this work we consider only warped spacetimes of Type I, in other words α is Lorentzian and γ is Riemannian.

Respective dimensions of V_1, V_2 are p and q . Geometric elements associated with V_1, V_2 are respectively labelled with indices 1, 2. For instance Δ_2 is the Laplace-Beltrami operator corresponding to (V_2, γ) , etc. Indices A, B label coordinates in V_1 (resp. i, j in V_2). For x and y in $V_1 \times V_2$, instead of the canonical co-ordinates x^A, x^i we can use the intrinsic notation

$$x = (u, \xi), \quad y = (v, \eta) \quad \text{with } u, v \in V_1, \quad \xi, \eta \in V_2.$$

The warping is just a geometric structure; it arises in a lot of different physical situations. For instance the FRW universe of cosmology (and its anisotropic generalizations), any metric with spherical symmetry (stationary or not), and also

the bulk spacetimes of brane theory, are warped. The most simple nontrivial and nondegenerate example of warping is given by the Friedman-Robertson-Walker line element.

Let us summarize the nice properties of warped spacetime:

- a) to a large extent, we control their isometries.
- b) the motion of a test particle in any warped spacetime admits a first integral of particular interest; this remark can be made already at the classical level. In its quantum mechanical version, it provides us with an observable which can be diagonalized together with the Klein-Gordon operator. This situation allows for a mode decomposition of the solutions to the KG equation.
- c) Each mode is a tensor product, which results in a separation of the variables; the skew-symmetric form defined on a given mode admits a factorization.

Unless otherwise specified, throughout this paper we make this technical assumption:

We assume that V_2 is compact and connected.

Remark. We are mainly interested in cases where spacetime is warped in a unique way, up to trivial re-definition. Definition. A trivial re-definition of the warped structure (6) consists in the replacement of S and γ respectively by

$$S' = \rho S, \quad \gamma' = \rho^{-1} \gamma$$

which results in the replacement of Δ_2 by $\Delta'_2 = \rho \Delta_2$ whereas all the eigenvalues are multiplied by ρ , say $\lambda'_n = \rho \lambda_n$. The equation (21) remains unchanged, for $\lambda'_n S'^{-1} = \lambda_n S^{-1}$. Moreover $\mathcal{S}_n, \mathcal{E}_n$ and the operator D remain unchanged.

3.1 Isometries of warped spacetimes

In this section we summarize relevant results of J. Carot and J. da Costa [11] and M. Sánchez [12]

The product structure of $V_1 \times V_2$ allows for the canonical decomposition of any vector field X into its first and second pieces, say $X = X_{(1)} + X_{(2)}$, with $X_{(1)} = X^A \partial_A$, $X_{(2)} = X^j \partial_j$.

Remark: This decomposition does not mean that $X_{(1)}$ (resp. $X_{(2)}$) might always be regarded as a vector field on V_1 (resp. V_2). But this situation happens in some particular cases of interest, see below. Besides, if we fix ξ (resp. u) we can regard $X_{(1)}$ (resp. $X_{(2)}$) as a vector field on $V_1 \times \{\xi\}$ (resp. $\{u\} \times V_2$).

We can distinguish three cases:

- i) First pure case: $X = X_{(1)}$, $X_{(2)} = 0$
- ii) Second pure case: $X = X_{(2)}$, $X_{(1)} = 0$
- iii) Mixed case: X has both pieces nonvanishing.

The following result has been proved in references [11] [12].

Theorem 0 (Carot and da Costa 1993, M. Sánchez 1998).

In the first pure case, X is Killing for (V, g) iff $X_{(1)}$ is Killing for (V_1, α) and in addition we have $X^A \partial_A S = 0$

In the second pure case, X is Killing for (V, g) iff $X_{(2)}$ is Killing for (V_2, γ)

In general, If X is Killing for (V, g) , then $X_{(1)}$ is Killing for $(V_1, \alpha) \times \{\xi\}$, $\forall \xi \in V_2$, and $X_{(2)}$ is a Conformally Killing vector for $\{u\} \times (V_2, \gamma)$, $\forall u \in V_1$. (the converse may not be true!)

In fact these results can be directly derived from the formula (6) if we consider α and γ as particular tensor fields on (V, g) . Hint: split the Lie derivative operator as $L_X = L_1 + L_2$ where 1, 2 refer to the vector fields $X_{(1)}, X_{(2)}$ respectively. Observe that the well-known general formula

$$L_Z t_{\mu\nu} = Z^\sigma \partial_\sigma t_{\mu\nu} + t_{\sigma\nu} \partial_\mu Z^\sigma + t_{\mu\sigma} \partial_\nu Z^\sigma \quad (7)$$

entails

$$L_1 \gamma = L_2 \alpha = 0 \quad (8)$$

We obtain that X is Killing for (V, g) when

$$L_1 \alpha - (L_1 S) \gamma - S L_2 \gamma = 0 \quad (9)$$

Note that $(L_1 \alpha)_{ij}$ and $(L_2 \gamma)_{AB}$ always vanish. In contradistinction $(L_1 \alpha)_{Ai}$ and $(L_2 \gamma)_{Ai}$ may be different from zero. Actually, it stems from (7) that

$$(L_1 \alpha)_{Ai} = \alpha_{A\sigma} \partial_i X_{(1)}^\sigma + \alpha_{\sigma i} \partial_A X_{(1)}^\sigma$$

$$(L_1 \alpha)_{Ai} = \alpha_{AB} \partial_i X_{(1)}^B + \alpha_{Bi} \partial_A X_{(1)}^B$$

But α_{Bi} is zero, thus

$$(L_1 \alpha)_{Ai} = \alpha_{AB} \partial_i X_{(1)}^B \quad (10)$$

and similarly

$$(L_2 \gamma)_{Ai} = \gamma_{ji} \partial_A X_{(2)}^j \quad (11)$$

So

$$(L_1 \alpha)_{ij} = 0 \quad (12)$$

$$(L_2 \gamma)_{ij} = -(L_1 \log S) \gamma_{ij} \quad (13)$$

In the first pure case, equation (9) reduces to $L_1 \alpha = (L_1 S) \gamma$. Taking the ij components of this formula yields $L_1 S = 0$, which entails $L_1 \alpha = 0$. Thus in particular $(L_1 \alpha)_{Ai} = 0$. As α_{AB} is invertible, (10) tells that X^B depends only on the variables x^A . So X is a vector field on (V_1, α) .

In the second pure case, it is clear that $L_2 \gamma = 0$. We can write in particular $(L_2 \gamma)_{Ai} = 0$. As γ_{ij} is invertible, (10) tells that X^j depends only on the variables x^i . So X is a vector field on (V_2, γ) .

In the mixed case, $X_{(1)}$ (resp. $X_{(2)}$) may also depend on x^i (resp. x^A). Still, taking the AB components of (9) yields $(L_1 \alpha)_{AB} = 0$, in other words

$$X_{(1)}^C \partial_C \alpha_{AB} + \alpha_{AC} \partial_B X_{(1)}^C + \alpha_{CB} \partial_A X_{(1)}^C = 0$$

so, for fixed ξ with coordinates x^i in V_2 , $X_{(1)}^C$ is Killing of (V_1, α) . Taking the ij components of (9) yields

$$(L_2 \gamma)_{ij} = -(L_1 \log S) \gamma_{ij} \quad (14)$$

So, if we ignore its dependence on the coordinates x^A , we can assert that $X_{(2)}$ is conformally Killing for the metric γ_{ij} . Multiplication of the above formula by γ^{ij} and contraction of the indices provide

$$\gamma^{ij}(\mathcal{L}_2\gamma)_{ij} = -q\mathcal{L}_1 \log S \quad (15)$$

Remark: This formula is (trivially) satisfied also in the first pure case.

In view of these results, we could say, loosely speaking, that in practice "most" Killing vectors are inherited from the symmetries of the second factor manifold, whereas the occurrence of the first pure and mixed cases is somehow "exceptional". This point can be made more precise, as follows.

In the second pure case, we observe that $X = X_{(2)}$ is Killing for (V_2, γ) irrespective of the warping factor. In this case, keeping X, α, γ fixed, we can arbitrarily replace S by another positive function S' . It is obvious that X remains Killing for the metric $g' = \alpha \oplus (-S')\gamma$.

In contrast, in the 1st pure case, $X = X_{(1)}$ may remain Killing for g' only if $\mathcal{L}_1 S'$ vanishes, and in the mixed case, X may remain Killing for g' only if S' satisfies the necessary condition (15).

Definitions.

A Killing vector of (V, g) will be called *structural* when it is the lift of a Killing vector of (V_2, γ) , say $X = X_{(2)}$.

Remark. Structural isometries are preserved, if we arbitrarily modify the warping factor, leaving the factor-metrics α and γ unchanged.

A Killing vector X of (V, g) will be called *factorial* when $X_{(1)}$ is either zero or Killing for (V_1, α) whereas $X_{(2)}$ is either zero or Killing for (V_2, γ)

A Killing vector X of (V, g) will be called *extraordinary* when $X_{(2)}$ is a *properly* conformal vector of $u \times (V_2, \gamma)$ for all $u \in V_1$ (by properly conformal we mean that it is not an isometry).

The following statements can be read off from Theorem 0.

Corollary

When (V_1, α) has no Killing vector, then every Killing vector of (V, g) is structural.

When (V_1, α) admits infinitesimal isometries, the existence of a non-structural symmetry in (V, g) still requires a particular shape of the warping factor, expressed by formula (15).

The definitions given above help to classify the isometries as: structural, factorial, extraordinary. In this paper we limit our investigation to warped spacetimes that are free of extraordinary Killing vector.

Under this restriction, in view of Theorem 0, we can assert that if X is Killing for V , we simply have $X = X_{(1)} + X_{(2)}$ where, on the one hand $X_{(1)}$ is Killing for (V_1, α) and in addition $X_{(1)}^A \partial_A S = 0$. On the other hand $X_{(2)}$ is Killing for (V_2, γ) .

In general it remains possible to have a Killing vector X such that $X_{(1)}^A \partial_A S$ vanishes. We shall only consider the cases where X is

either a) always timelike or b) always spacelike.

Special cases

$$p = 1$$

This case describes a generalized FRW spacetime, one has $V_1 \subset \mathbf{R}$. Existence of a non-structural Killing vector for (V, g) , requires an exceptional shape of the warping factor.

The exceptional warping factors are determined by an ordinary differential equation $\ddot{\Theta} \exp(2\Theta) = \text{const.}$ in terms of a suitable time scale t . They are listed in the literature and include, of course, the de Sitter universe. In fact the de Sitter metric is *a priori* exceptional because its full isometry group is larger than the isometry group of one of its space sections.

$$p = 2$$

Assuming that the lines defined in V_1 by $S = \text{const.}$ are everywhere timelike (resp. spacelike). This assumption excludes the trivial case of a constant warping factor. The shape of the warping factor induces on (V_1, α) a preferred net of orthogonal coordinate lines, and a preferred foliation by Cauchy "surfaces" which are in fact the lines $S = \text{const.}$ (resp. their orthogonal trajectories).

Four-dimensional spacetime $p = q = 2$

This situation is referred to as Class B in ref. [11]. It encompasses all spacetimes with spherical symmetry.

4 Mode Decomposition

4.1 An interesting constant of the motion

At the classical level, it is remarkable that the phase-space function $2K_{\text{cl}} = \gamma^{AB} p_A p_B$ is invariant by action of the geodesic flow. The quantum mechanical version of this result states that the differential operator $2K_{\text{qu}} = -\Delta_2$ commutes [16] with the KG operator for minimal coupling, namely $\nabla^2 + m^2$. Indeed, the structure of warped spacetime allows to derive a useful identity,

$$\nabla^2 \Psi = \Delta_1^\# \Psi - S^{-1} \Delta_2 \Psi \quad (16)$$

where Δ_2 is the q -dimensional Laplacian associated with the manifold (V_2, γ) and where we define in (V_1, α) the "warped Laplacian" of a function

$$\Delta_1^\# \Psi = \frac{1}{\sqrt{|\alpha|}} S^{-q/2} \partial_A (\sqrt{|\alpha|} S^{q/2} g^{AB} \partial_B \Psi)$$

which can be re-arranged as

$$\Delta_1^\# \Psi = \Delta_1 \Psi + \frac{1}{2} q \alpha^{AB} (\partial_A \log S) \partial_B \Psi \quad (17)$$

irrespective of whether Ψ is a solution to (1) or not.

Note that $g^{AB} = \alpha^{AB}$, thus the second order differential operator Δ_1^\sharp only affects quantities depending on the x^A variables. In contrast, Δ_2 , as an operator extended to functions on V , does not affect the quantities depending on x^A only.

Examples

When (V, g) is FRW with flat space sections, (V_2, γ) is the three-dimensional plane, and K/m is the kinetic energy. In this particular case, conservation of K is trivial: space translation invariance implies that each p_k is conserved. For FRW with spherical space sections, (V_2, γ) is a three-dimensional sphere, in principle, conservation of K could be also derived from the constants of the motion associated with the Killing vectors on the sphere. When (V, g) is some inhomogeneous and anisotropic generalization of FRW, (V_2, γ) has perhaps no isometry at all, but K is still conserved. In that case, it is natural to consider K/m as a generalization of the kinetic energy.

When (V, g) is four-dimensional with spherical symmetry, (V_2, γ) is the sphere with unit radius and line element

$$\gamma_{ij}dx^i dx^j = d\theta^2 + \sin^2 \theta d\varphi^2$$

We find, with the standard notation, that

$$2K_{\text{qu}} = \Delta_2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + (\frac{1}{\sin \theta} \frac{\partial}{\partial \varphi})^2$$

which is opposite to the square of angular momentum.

These examples indicate that, in most cases of interest, K has a natural physical meaning.

4.2 Mode Solutions, Separation of Variables

For all $\lambda \in \text{Spec} V_2$ the *Mode Space* associated with λ is by definition the linear space $\mathcal{H}[\lambda]$ made of solutions to (1) that are also eigenfunctions of $-\Delta_2$ with eigenvalue λ . Since we assume that V_2 is compact and connected, $\text{Spec}(V_2)$ is a discrete sequence

$$0 < \lambda_1 < \lambda_2 < \dots$$

We simply write \mathcal{H}_n as a short-hand for $\mathcal{H}[\lambda_n]$. This terminology agrees with the usual one in the special case of FRW spacetimes.

Let us now turn to features associated with (V_2, γ) . The coordinates x^A are ignorable when Δ_2 acts on Φ , so let $\mathcal{E}[\lambda]$ be the (complex) eigenspace of $-\Delta_2$ in $C^\infty(V_2, \gamma)$. We write \mathcal{E}_n as a short-hand for $\mathcal{E}[\lambda_n]$.

Note that \mathcal{E} has finite dimension $r(n)$ and admits a real basis [13].

Since (V_2, γ) is elliptic, $C^\infty(V_2)$ (thus also \mathcal{E}_n) is endowed with the *positive definite* scalar product

$$\langle F, H \rangle_2 = \int_{V_2} F^* H \sqrt{\gamma} d_2^g x \quad (18)$$

For F and H real, $\langle F, H \rangle_2$ is a positive quadratic form.

Let $E_{1,n}, \dots, E_{a,n}, \dots, E_{r,n}$ be a real orthonormal basis of \mathcal{E}_n , thus $\langle E_a, E_b \rangle_2 = \delta_{ab}$.

We focus of these solutions to (1) that are a *finite* sum of modes, say

$$\phi = \sum \Phi_n \quad (19)$$

The loss of generality resulting from this restriction to finite sums will, to a large extent, be compensated when ultimately performing the Hilbert completion of the positive-frequency sector.

It is important to check if the modes that we have defined actually are orthogonal. So, the question arises as to know whether \mathcal{H}_n is orthogonal to \mathcal{H}_l for $l \neq n$, in the sense of the sesquilinear form associated with the Gordon current [16]. In fact the answer is: yes (see [16] Section 5, Proposition 5).

As long as λ_n is kept fixed, the label n referring to a determined eigenvalue of K can be provisionally dropped. Thus Φ stands for Φ_n in Mode n . For some nonnegative $\lambda \in \text{Spec}(V_2)$ we have

$$\Delta_2 \Phi = -\lambda \Phi \quad (20)$$

and (1) reduces to

$$(\Delta_1^\sharp + \lambda S^{-1} + m^2) \Phi = 0 \quad (21)$$

The coordinates x^k are ignorable when Δ_1^\sharp acts on Φ , so the equation above is a p -dimensional problem. Let \mathcal{S} (resp. \mathcal{S}^R) be the linear space of C^∞ complex-valued (resp. real-valued) functions $f(x^A)$ satisfying the equation

$$(\Delta_1^\sharp + \lambda S^{-1} + m^2) f = 0 \quad (22)$$

We have shown [16] that (22) can be cast into the form

$$(\Delta_1 + \Xi) \hat{f} = 0 \quad (23)$$

where we have set

$$\hat{f} = S^{q/4} f \quad (24)$$

$$\Xi = (S^{q/4} \Delta_1 S^{-q/4}) - \frac{q^2}{8} S^{-2} \alpha(\partial S, \partial S) + \lambda S^{-1} + m^2 \quad (25)$$

On the one hand (23) is simpler than the original equation (1) because it is only a p -dimensional problem; on the other hand it seems to be a little more complicated, for it involves a "source term" Ξ .

Fortunately, the KG equation with a source term still admits a conserved current¹, which entails that the quantity

$$\sigma(\hat{f}, \hat{h}) = \int_L (\hat{f} \alpha^{AB} \partial_B \hat{h} - \hat{h} \alpha^{AB} \partial_B \hat{f}) \quad dL_A \quad (26)$$

if it is finite, doesnot depend on the spacelike $p - 1$ dimensional surface L . Let us ensure that σ actually is finite; when L is not compact it becomes necessary that the functions f and h decrease rapidly enough at spatial infinity. In order to be more precise, we make either one of the following assumptions:

¹Notice a slight change of notation with respect to ref. [16], because here J denotes the complex structure.

Assumption 1

For $p > 1$, we suppose $(V_1, \alpha) \approx \mathbf{R} \times \text{compact}$.

Assumption 2

For $p > 1$, we suppose that (V_1, α) is globally hyperbolic, which implies some Cauchy surface $L \in V_1$. Moreover we include in the definition of \mathcal{S} the condition that f in equation (22) arises from initial data with compact support.²

Note that none of these assumptions contains the other one, although they overlap. From now on, we suppose that one of them is realized. Only if necessary, shall we specify which one.

Remark:

Strictly speaking, spherically symmetric models that arise in astrophysics fail to satisfy these assumptions, for they have $V_1 \approx \mathbf{R} \times \mathbf{R}^+$. But it is possible to enlarge the present framework with help of a prescription: extending the Cauchy "surface" $t = 0$ to negative values of r and making the convention that $f(t, -r) = f(t, r)$.

Now we can endow \mathcal{S}_n^R with the skew-symmetric form

$$W(f, h) = \sigma(\hat{f}, \hat{h}) \quad (27)$$

With help of (24) we can check that

$$W(f, h) = \int_L S^{q/2} (f \alpha^{AB} \partial_B h - h \alpha^{AB} \partial_B f) dL_A \quad (28)$$

as we had written in ref. [16]. Extending W to the complex domain, we define a sesquilinear form

$$(f; h)_1 = -iW(f^*, h) \quad (29)$$

Remark

When $p = 1$ then W reduces to the Wronskian of two functions of a single variable x^0 . This situation occurs in generalized FRW spacetimes.

The general solution to (20)(21) takes on the form

$$\Phi = \sum_1^{r(n)} f_a(x^A) E_a(x^j) \quad (30)$$

where every f_a is a smooth function on V_1 , satisfying equation (22), and the mode label n has been dropped from $E_{a,n}$. In other words we can write

$$\mathcal{H}_n = \mathcal{S}_n \otimes \mathcal{E}_n \quad (31)$$

It is clear that the sesquilinear form $(\Phi; \Omega)$ and the skew-symmetric form ϖ can be restricted to \mathcal{H}_n .

Definition

Let w be the restriction of ϖ to \mathcal{H}_n .

²This property, when it holds, is independent of the choice of the Cauchy surface (see Ref. [1] pp. 53-58).

Theorem 1 *If (V_1, α) is globally hyperbolic, then all the solutions of the form (19) are uniquely determined by their Cauchy data on $L \times V_2$ where L is a Cauchy surface in (V_1, α) .*

The proof is mode-wise. For Φ in mode n , we write (30) and observe that

$$\Phi_n|_\Sigma = \sum_a f_a(u)|_L E_a(\xi)$$

$$(\partial_0 \Phi_n)|_\Sigma = \sum_a (\partial f_a)|_L E_a(\xi)$$

But \hat{f}_a is solution to a reduced KG equation (23); it follows that \hat{f} and therefore f , is uniquely determined by its Cauchy data on L .

At this stage one may be tempted to claim that $L \times V_2$ is a Cauchy surface for (V, g) . But we leave this question open.

For complex functions in \mathcal{H}_n we can write in general

$$(\Phi; \Psi) = -iw(\Phi^*, \Psi), \quad (f; h)_1 = -iW(f^*, h) \quad (32)$$

4.3 Factorization

Let Φ and Ω be solutions in mode n . So Φ can be written as (30) and

$$\Omega = \sum_1^{r(n)} h_b E_b \quad (33)$$

Under reasonable assumptions [16], the sesquilinear form w on \mathcal{H}_n is *compatible* with the structure of tensorial product. Indeed for $\Phi = fF \in \mathcal{H}_n$ and $\Omega = hH \in \mathcal{H}_n$ with $f, h \in \mathcal{S}_n$ and $F, H \in \mathcal{E}_n$, we have proved that

$$(\Phi; \Omega) = (f; h)_1 < F, H >_2 \quad (34)$$

where $< F, H >_2$ is invariant under the symmetries of (V_2, γ) .

In order to ensure that $(f; h)_1$ is finite, in ref. [16] we used Assumption I. But the reader will easily check that the derivation of (34) and the conclusions of Section 5 in ref. [16] remain valid as well under Assumption II.

Equation (34) can be reformulated in terms of the underlying skew-symmetric forms. For Φ and Ω as above, then (32) entails that

$$(\Phi; \Omega) = -iw(\Phi^*; \Omega)$$

Our result of [16] tells that

$$w(\Phi, \Omega) = W(f, h) < F, H >_2 \quad (35)$$

Here we specify $F = E_a$, $H = E_b$ and obtain that

$$(\Phi; \Omega) = \sum (f_a; h_b)_1 \delta_{ab}$$

In view of equations (32) we can re-write this in terms of skew-symmetric forms, say

$$w(\Phi^*, \Omega) = \sum_{ab} W(f_a^*, h_b) < E_a, E_b >_2 \quad (36)$$

(For each mode \mathcal{H}_n) it is now clear that w is the skew-symmetric form induced by W on \mathcal{H}_n according to the tensorial product structure of it.

4.4 Modes, Complex Structure and Invariant Splitting

We turn back to the problem of determining a complex structure J in the full space of solutions to (1). If we succeed, the positive (resp. negative)-frequency part of \mathcal{K} , say \mathcal{K}^+ (resp. \mathcal{K}^-) is made of solutions satisfying $\Pi^+\phi = \phi$ (resp. $\Pi^-\phi = \phi$).

Insofar as the quantity K distinguished in Section 4.1 has a physical meaning, it should be represented by an operator which maps \mathcal{K}^+ into itself. This condition can be formulated as follows:

For all $\phi \in \mathcal{K}$, we have $\Pi^+\phi \in \mathcal{K}^+$. Now, $K(\Pi^+\phi)$ must enjoy the same property, thus $\Pi^+K\Pi^+ = K\Pi^+$. But Π^+ is idempotent, so we can write $\Pi^+K\Pi^+ = (K\Pi^+)\Pi^+$, in other words

$$[\Pi^+, K\Pi^+] = 0 \quad (37)$$

In order to preserve the symmetry between positive and negative frequencies, it is natural to require as well

$$[\Pi^-, K\Pi^-] = 0 \quad (38)$$

But $\Pi^- = 1 - \Pi^+$. Inserting this relation into (37) (38) finally yields $[K, \Pi^+] = 0$. It is now clear that K leaves stable both \mathcal{K}^\pm iff K commutes with Π^+ or equivalently with J . In order to implement this condition, the separation of frequencies will be mode-wise carried out. We are led to focus on the solutions to (1) that can be developed in modes, like ϕ in (19). Let \mathcal{L} be the linear space formed by these solutions.

It is a mere exercise to check that, provided J_n is a complex structure in \mathcal{H}^R , and J_n is positive with respect to w , then the operator J defined by direct sum (say $J = \bigoplus J_n$ that is $J\phi = \sum J_n\Phi_n$) is a complex structure in \mathcal{L}^R and is positive with respect to ϖ .

Now, in each shell \mathcal{H}_n , we look for suitable projectors Π_n^\pm , or equivalently we look for a complex structure J_n acting in $\mathcal{S}_n \otimes \mathcal{E}_n$ as a linear operator; let us formulate:

Problem at a fixed Mode n

Find a complex structure operator J_n acting in \mathcal{H}_n , positive with respect to w and invariant under the structural isometries.

For product solutions structural isometries act as follows

$$T(f(x^A)F(x^j)) = f(x^A)TF(x^j)$$

where T is an isometric transformation of (V_2, γ) .

In order to solve the above problem, we shall resort to a few results concerning complex structures in tensor-product spaces. They are displayed in the next Section.

5 Skew-symmetric forms and complex structure in a tensor product

Definition A *complex structure* on a *real* linear space \mathcal{A} is a linear operator J such that $J^2 = -1$.

In this section, Φ, Ψ, Ω may belong to \mathcal{A} or its complexified, \mathcal{A}^C .

When \mathcal{A} is endowed with a skew-symmetric form w , it is noteworthy that J leaves w invariant iff $w(\Phi, J\Psi)$ is symmetric under exchange of Φ with Ψ .

Now, consider *real* vector spaces $\mathcal{A}_1, \mathcal{A}_2$. Let $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, where $\dim \mathcal{A}_2 < \infty$. We assume that \mathcal{A}_1 is endowed with a *skew-symmetric form* $W(f, h)$ whereas \mathcal{A}_2 is endowed with a *quadratic form* $Q(F, H)$.

It follows from our assumptions that \mathcal{A} is in turn endowed with the unique skew-symmetric form $w(\Phi; \Omega)$ such that, if $\Phi = f \otimes F$ and $\Omega = h \otimes H$ are in \mathcal{A} we have

$$w(\Phi, \Omega) = W(f, h) Q(F, H) \quad (39)$$

We can say, in an obvious sense, that this skew form is *compatible* with the tensorial product.

In addition we assume that the quadratic form Q is *positive definite* and we adopt this notation

$$Q(F, H) = \langle F, H \rangle$$

Since \mathcal{A}_2 has a finite dimension, it admits an orthonormal basis, say $\{E_a\}$, with $a = 1, \dots, \dim \mathcal{A}_2$. It is clear that

$$Q(E_a, E_b) = \langle E_a, E_b \rangle = \delta_{ab}$$

Under these assumptions we can easily check that

Proposition 1 Any complex structure operator, say J_1 , defined on \mathcal{A}_1 induces a complex structure $J = J_1 \otimes 1$ defined on \mathcal{A} . If J_1 leaves W invariant, then J leaves w invariant.

Proof: As J is characterized by $J(fF) = (J_1 f)F$ it is obvious that $J^2 = -1$.

Invariance of w can be first proved for products. In this case, $w(\Phi, \Omega)$ is given by (39) and we have $J\Phi = (J_1 f)F$, $J\Omega = (J_1 h)H$. Therefore

$$w(J\Phi, J\Omega) = W(J_1 f, J_1 h) Q(F, H) \quad (40)$$

Since we assume that J_1 leaves W invariant, we can replace $W(J_1 f, J_1 h)$ by simply $W(f, h)$, thus $w(J\Phi, J\Omega) = w(\Phi, \Omega)$ when Φ, Ω are products.

But in general we must write

$$\Phi = \sum_a f_a E_a \quad \Omega = \sum_b h_b E_b \quad (41)$$

So $w(\Phi, \Omega) = \sum_{a,b} w(f_a E_a, h_b E_b)$. Apply formula (39) with $F = E_a, H = E_b$. We obtain

$$w(\Phi, \Omega) = \sum_{a,b} W(f_a, h_b) Q(E_a, E_b) \quad (42)$$

Since $\{E_a\}$ is orthonormal, it follows that

$$w(\Phi, \Omega) = \sum_a W(f_a, h_a) \quad (43)$$

On the other hand, $J\Phi = \sum_a J(f_a E_a) = \sum_a (J_1 f_a) E_a$, whereas $J\Omega = \sum_b J(h_b E_b) = \sum_b (J_1 h_b) E_b$. Hence

$$w(J\Phi, J\Omega) = \sum W(J_1 f_a, J_1 h_b) Q(E_a, E_b) = \sum W(f_a, h_b) \delta_{ab} \quad (44)$$

which is nothing but expression (43).

Definition

Let J be a complex structure on a linear space \mathcal{A} endowed with the skew-symmetric form w . We say that J is *positive* with respect to w when it leaves w invariant and satisfies $w(\Phi, J\Phi) > 0$ for every $\Phi \neq 0$.

Proposition 2 *If J_1 is positive with respect to W , then J is positive with respect to w .*

Proof

We know from Proposition 1 that J is a complex structure for \mathcal{A} and leaves w invariant. Now let us evaluate $w(\Phi, J\Phi)$. For any Φ in $\mathcal{A}_1 \otimes \mathcal{A}_2$ we can write

$$\Phi = \sum f_a E_a \quad J\Phi = \sum (J_1 f_a) E_a \quad (45)$$

where $\langle E_a, E_b \rangle = \delta_{ab}$.

$$w(\Phi, J\Phi) = \sum w(f_a E_a, (J_1 f_b) E_b) = \sum W(f_a, J_1 f_b) \langle E_a, E_b \rangle$$

But J_1 is supposed to be positive with respect to W . Thus each term $W(f_a, J_1 f_a) > 0$ unless f_a vanishes. It follows that $w(\Phi, J\Phi) > 0$ unless f_a vanishes for all a , which would occur only when $\Phi = 0$.

Remark At this stage it is worthwhile recalling that the eigenspaces of Π^+ and Π^- are mutually orthogonal.

6 Invariant Separation of the Frequencies.

6.1 Structural Invariance

Let us now turn to the problem at mode n , formulated in Section 4. Let \mathcal{E}_n^R contain the *real* elements of \mathcal{E}_n . We claim that

Theorem 2 *Provided that $J_{1,n}$ is a complex structure in \mathcal{S}_n^R and is positive with respect to the skew form $W(f, h)$, then a solution to the Problem at Mode n is given by*

$$J_n = J_{1,n} \otimes I_{2,n} \quad (46)$$

where $I_{2,n}$ is the identity on \mathcal{E}_n^R .

Formula (46) will be referred to as a *canonical solution* to the problem at a fixed mode n . It means that for any basis in \mathcal{E}_n

$$J_n \sum f_a E_a = \sum ((J_{1,n} f_a) E_a$$

The proof is in three steps, one must prove that:

- i) J is a complex structure,
- ii) it is positive with respect to the skew form in \mathcal{H} ,
- iii) it is invariant under the continuous isometries of (V_2, γ) .

In view of (31), points (i) and (ii) stem from application of Propositions 1, 2 with $\mathcal{A}_1 = \mathcal{S}^R$ endowed with a skew form W as in (28) and $\mathcal{A}_2 = \mathcal{E}^R$ endowed with $<, >_2$ as in (18).

Finally, invariance under the isometries of (V_2, γ) is obvious, for these transformations affect neither the functions of x^A nor the identity on \mathcal{E} . This achieves the proof.

By Theorem 2, the initial problem, formulated in \mathcal{K} and involving a symmetry requirement, has been reduced to the question as to construct in each \mathcal{S}_n a complex structure which is positive with respect to W . This reduced problem does not involve any symmetry condition and is posed in a lower dimension (p instead of $p + q$). In the special case where $p = 1$, it is easy to find the complex structure, because \mathcal{S} is two-dimensional. Otherwise, the issue seems a bit more difficult, because \mathcal{S} has in general infinitely many dimensions and is defined through a KG equation with a source term, Ξ .

Let us now prove that a suitable $J_{1,n}$ as invoked in the theorem above actually exists. Our basic tool is the observation that, in a globally hyperbolic spacetime, the KG equation with a source term has a well-posed initial value formulation (see [15], see [1] p. 56).

\mathcal{S}_n^R is defined through equation (22) or equivalently (23). In (23) the source term Ξ_n , explicitly given by (25), is a smooth function on V_1 . Since (V_1, α) is globally hyperbolic, we can exhibit an operator $J_{1,n}$ enjoying the required properties, with help of a Cauchy surface in (V_1, α) . We proceed as follows: Let $\hat{\mathcal{S}}_n$ be the vector space of real solutions to (23). Let L be a Cauchy surface (V_1, α) . By the procedure indicated in Appendix we construct a complex-structure operator j_n acting in $\hat{\mathcal{S}}_n^R$. This j_n is positive with respect to $\sigma(\hat{f}, \hat{h})$. In view of (27) it is a mere exercise to check that

$$J_{1,n} = S^{-q/4} j_n S^{q/4}$$

is a complex-structure operator acting in \mathcal{S}_n^R and is positive with respect to $W(f, h)$.

After having ensured existence of J_n for all n , we are now in a position to define the subspaces \mathcal{S}_n^\pm .

Notation We define the projectors $\Pi_{1,n}^\pm = \frac{1}{2}(1 \pm iJ_{1,n})$.

Let $f \in \mathcal{S}_n^\pm$ when $f \in \mathcal{S}_n$ and $\Pi_{1,n}^\pm f = f$.

It is clear that $\mathcal{S}_n = \mathcal{S}_n^+ \oplus \mathcal{S}_n^-$. Moreover \mathcal{S}_n^+ and \mathcal{S}_n^- are mutually orthogonal in $(\cdot, \cdot)_1$.

Similarly we define

$$\mathcal{H}_n^\pm = \mathcal{S}_n^\pm \otimes \mathcal{E}_n$$

we obtain $\mathcal{H}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$. and we can say $\Phi \in \mathcal{H}_n^\pm$ iff $\Phi \in \mathcal{H}_n$ and $\Pi_n^\pm \Phi = \Phi$. Let us now achieve our goal, considering generic solutions to (1) in the form of finite sums like $\Phi = \sum \Phi_n$, $\Phi_n \in \mathcal{H}_n$. In other words $\Phi \in \mathcal{H}$ where $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$. From J_n and Π_n^\pm , by direct sum, we construct J and Π^\pm acting in the full space \mathcal{H} , according to

$$J \sum \Phi_n = \sum J_n \Phi_n, \quad \Pi^\pm \sum \Phi_n = \sum \Pi_n^\pm \Phi_n$$

We end up with $\mathcal{H} = \mathcal{H}^+ + \mathcal{H}^-$ where $\mathcal{H}^\pm = \bigoplus \mathcal{H}_n^\pm$, and \mathcal{H}^+ orthogonal to \mathcal{H}^- . Here, \mathcal{H} endowed with $(\cdot; \cdot)$ is only pre-Hilbert. Its completion will provide a Hilbert space of one-particle states with positive frequency. We summarize

Proposition 3 *Provided that (V_1, α) is globally hyperbolic, there exists a complex-structure operator J which is positive with respect to the skew form (2) and commutes with structural isometries.*

In other words, a splitting of the solutions to equation (1) according to positive and negative frequencies is possible and invariant under structural isometries.

6.2 Non-structural Isometries

The previous result is satisfactory insofar as (V_1, α) has no Killing vector. But our purpose was to investigate a little further. So, let us consider the cases referred to as a) and b) in Section 3.

a) $X = X_{(1)}$ globally timelike, (V_1, α) is stationary.

There are coordinates where $X^A \partial_A = \partial_0$. But there is no evidence that the operator $J_{1,n}$ build as in the previous subsection would be invariant under this X . Under Assumption I, we propose an alternative choice, more direct and more natural. Every f in \mathcal{S}_n will be developed as

$$f = \int_{-\infty}^{+\infty} f_E dE \quad (47)$$

where

$$Xf = iEf_E \quad (48)$$

We define

$$\Pi_{1,n}^+ f = \int_0^{+\infty} f_E dE \quad (49)$$

and similarly $\Pi_{1,n}^-$. It follows that $\Pi_{1,n}^+$ commutes with ∂_0 . In agreement with this modification, we propose the complex structure defined by $J_{1,n} = -i(2\Pi_{1,n}^+ - 1)$. It is easy to verify that this operator leaves W invariant and is positive with respect to W . Hint: $[\Pi_{1,n}^+, X] = 0$ and $X\hat{f} = S^{q/4}Xf$. From the new $J_{1,n}$ the formula of Theorem 2 yields a new J_n which remains of course invariant under structural isometries, but is additionally invariant under the time translations.

b) $X = X_{(1)}$ globally spacelike.

All we need is to manage that the operator $J_{1,n}$ be invariant under $X = X_{(1)}$. It would be sufficient that, in the procedure described in Appendix, we chose L to be invariant by action of X . Still the question arises: does it exist in (V_1, α) a Cauchy surface invariant by the isometries of this manifold? Naturally, this question is more easy to handle in two dimensions.

Remark:

For applications, the case where $p = 2$ is of particular interest. Indeed every two-dimensional spacetime is locally conformal to Minkowski; thus whenever (V_1, α) is globally conformal to a two-dimensional Minkowski space, it is itself globally hyperbolic³.

7 Kernels

Consider a functional space \mathcal{F} endowed with a sesquilinear form noted $(.;.)$. Let A be a linear operator mapping \mathcal{F} into itself. We say that A has a kernel N (with respect to the form) when there exists in $\mathcal{F} \otimes \mathcal{F}$ a two-point function $N(x, y)$ such that $\forall \Phi \in \mathcal{F}$

$$(A\Phi)(y) = (N(y, x); \Phi(x)) \quad (50)$$

where $N_y(x)$ is $\forall y$ an element of \mathcal{F} . In fact N is a bi-scalar. When A is the identity, we say that N is a *reproducing kernel* [17] in \mathcal{F} .

Exemple: $\mathcal{F} = \mathcal{E}_n$, eigenspace of the Laplacian in (V_2, γ) , for a fixed eigenvalue n . In this example, the sesquilinear form is given by (18) and \mathcal{F} is a Hilbert space. In \mathcal{E}_n the identity admits a kernel

$$\Gamma_n(\xi, \eta) = \sum_{a=1}^r E_{a,n}(\xi) E_{a,n}(\eta) \quad (51)$$

We have $\langle \Gamma(\eta, \xi), F(\xi) \rangle_2 = F(\eta)$ for all $F \in \mathcal{E}_n$, thus Γ is a reproducing kernel on \mathcal{E}_n . In this case, note that $\Gamma(\eta, \xi)$ actually belongs to \mathcal{F} as a function of ξ labelled with η (and *vice versa*). Note that Γ is real and doesnot depend on the choice of a real orthonormal basis in \mathcal{E}_n . It is intrinsically determined by the geometry of (V_2, γ) [13]. In this example, Γ is unique because $\langle ., . \rangle_2$ is positive definite.

In fact, requiring that always $N \in \mathcal{F} \otimes \mathcal{F}$ would be too restrictive for the needs of quantum mechanics, therefore the kernels that are commonly considered are bi-scalar distributions, say $N \in \mathcal{F}' \otimes \mathcal{F}'$ where \mathcal{F}' is a suitable space of distributions.

Going back to the problem of positive/negative frequency splitting, our interest is in the possibility for Π^\pm to admit a kernel. The positive/negative-frequency kernel is formally defined as a (continuous) linear functional by

$$(\Pi^\pm \psi)(y) = (D^\pm(y, x); \psi(x)) \quad (52)$$

This functional is actually continuous, because Π^\pm is continuous in the sense of some Sobolev-space topology defined on the initial data. This point stems from the

³Global hyperbolicity only involves the causal structure; it is preserved by a conformal factor.

fact that (considering ψ as defined by initial data on some Cauchy surface) ψ varies continuously with the initial data (Ref. [1] p.56) and because the recipe given in Appendix 1 does not break this continuity.

But we proceed mode-wise thus, in each mode \mathcal{H}_n , it is natural to consider a kernel for Π_n^\pm . We take advantage of the factorization as follows; let $D_{1,n}^\pm$ be a kernel for $\Pi_{1,n}^\pm$ that is to say, $\forall f \in \mathcal{S}_n$

$$(f^\pm)(v) = (D_{1,n}^\pm(v, u); f(u))_1 \quad (53)$$

where $f^\pm \equiv \Pi_{1,n}^\pm f$ by definition.

Now it is not difficult to prove that

Proposition 4 *The bi-scalar*

$$D_n^+(y, x) = D_{1,n}^+(v, u) \Gamma_n(\eta, \xi) \quad (54)$$

is a kernel for Π_n^+ . It is manifestly invariant under structural isometries. Since $D_{1,n}^+(u, v)$ satisfies (22) in its argument u , then D_n^+ satisfies (1).

Proof:

Invariance under structural isometries stems from the following observations: Γ is the unique kernel of the identity in \mathcal{E}_n . The well-known isometric invariance of the Laplacian Δ_2 entails that each eigenspace \mathcal{E}_n is (globally) invariant under the isometries of (V_2, γ) . If T is such an isometry, it leaves invariant the q -dimensional scalar product $\langle F, G \rangle_2$.

If F is a function on (V_2, γ) we can write $TF = F(T\xi)$. Thus $E_{1,n}(u, \xi) \cdots E_{r,n}(u, \xi)$ is another real orthogonal basis of \mathcal{E}_n . So $\Gamma_n(T\eta, T\xi) = \Gamma_n(\eta, \xi)$ and we can write

$$D_n^+(Ty, Tx) = D_n^+(y, x)$$

In other words, any isometry of the second factor manifold leaves D_n^+ invariant \square .

Expression (54) will be called *canonical*.

The only arbitrariness involved in formula (54) is in the reduced kernel $D_{1,n}^+(v, u)$ which depends on the choice of a positive-energy projector in \mathcal{S}_n .

In general \mathcal{S} is an infinite dimensional vector space, with the exception that $\dim \mathcal{S} = 2$ when $\dim V_1 = 1$; this particular case has been described in [7].

Note that, when (V_2, γ) has constant curvature, explicit expressions for a basis of \mathcal{E}_n , hence for Γ_n , are available in closed form in the literature.

8 Concluding Remarks

With help of our mode decomposition, the problem of finding an invariant quantization of free particles in the $p + q$ -dimensional warped product $V_1 \times V_2$ has been reduced to a similar problem *without symmetry requirement*, but with a source term, in the p -dimensional manifold V_1 (this reduced problem being one-dimensional, in particular, when we start from a generalized FRW spacetime).

We have characterized a family of admissible complex-structure operators. Each one uniquely corresponds to a splitting of the solutions to the primitive problem into positive-frequency and negative-frequency parts. Our procedure respects isometric invariance, at least insofar as all the isometries of $V_1 \times V_2$ are induced by symmetries of its second factor (structural isometries). This case already encompasses a very large class of spacetimes.

When there are isometries induced by symmetries in the first factor manifold, the situation is still partially under control. For instance, the case where (V_1, α) is stationary can be handled, and gives rise to an operator J which commutes not only with structural isometries, but also with the time translations.

In contrast, there is no clue for the case where an extraordinary Killing field exists. Fortunately, the occurrence of this case is limited by the severe condition (14) involving the warping factor \sqrt{S} .

Note that our approach is concerned with one given structure of warped spacetime; it would become ambiguous in the degenerate cases where V can be regarded as warped in more than one manner. This remark applies to de Sitter space; fortunately, in that case, it is possible to construct an invariant vacuum by a different method [9]. In this paper we have considered spacetimes with smooth metrics; extension to more realistic situations requires further work.

The most general question as to know under which conditions the free motion of scalar particles in an "arbitrary spacetime" bearing Killing vectors admits an *isometrically invariant* quantization, remains open. However, isometric invariance is more easily implemented within the framework of warped spacetimes.

APPENDIX 1

Klein-Gordon equation with a "source term"

Consider N dimensional spacetime V_N , with coordinates x^0, x^1, \dots, x^{N-1} .

Consider the KG equation with a nonderivative external coupling

$$\nabla^2 \Phi + A(x) \Phi = 0 \quad (55)$$

where A is a smooth function.

For *real* solutions Φ, Ω the vector field $\Phi \nabla^\alpha \Omega - \Omega \nabla^\alpha \Phi$ is conserved. Under suitable technical assumptions the quantity

$$\varpi(\Phi, \Omega) = \int_\Sigma j \cdot d\Sigma = \int_\Sigma (\Phi \nabla^\alpha \Omega - \Omega \nabla^\alpha \Phi) d\Sigma_\alpha \quad (56)$$

is finite and doesnot depend on the choice of the spacelike hypersurface Σ . It defines a skew-symmetric form on the linear space of solutions to (55).

When Σ is defined by $x^0 = 0$ we have *on this hypersurface* $j \cdot \Sigma = j^0 d\Sigma_0$ hence

$$d\Sigma_0 = \sqrt{|g|} d^{N-1}x$$

If we can choose coordinates such that $g^{0\alpha}|_{\Sigma} = 0$ for $\alpha \neq 0$ we can simply write

$$\varpi(\Phi, \Omega) = \int_{x^0=0} (\Phi \partial_0 \Omega - \Omega \partial_0 \Phi) g^{00} \sqrt{|g|} d^{N-1}x \quad (57)$$

Let us stress that ϖ is intrinsically defined. In contradistinction, a complex structure operator J acting on the solutions to (55) is by no means unique. Such a J can be associated to each Cauchy surface. The recipe proposed below (valid in the presence of a source term) is inspired from, but not identical with, the procedure indicated by Ashtekar and Magnon in the context of minimal coupling. In addition, we shall apply this recipe, not in the full spacetime (V, g) , but within its first factor manifold, V_1 .

In order to build a complex structure we remind that each solution of (55) is uniquely and globally determined by its value and that of its time derivative on any Cauchy surface Σ , see p.56 of reference [1].

Therefore, the space of solutions to (55) is isomorphic to the vector space \mathcal{C}_{Σ} of couples

$$\begin{pmatrix} U \\ V \end{pmatrix}$$

where U and V are smooth functions on the Cauchy surface Σ , each solution Φ being represented by the couple

$$\begin{pmatrix} \Phi|_{\Sigma} \\ (\partial_0 \Phi)_{\Sigma} \end{pmatrix}$$

Naturally ϖ induces a skew form on these couples, denoted by the same typographic character, say

$$\varpi \left(\begin{pmatrix} U \\ V \end{pmatrix}, \begin{pmatrix} U' \\ V' \end{pmatrix} \right) = \int (UV' - U'V) g^{00} \sqrt{|g|} d^{N-1}x \quad (58)$$

Let J be defined by

$$J \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} -V \\ U \end{pmatrix} \quad (59)$$

Of course, J depends on Σ . According to our assumptions, it is clear that the new Cauchy data

$$(J\Phi)_{\Sigma} = -V, \quad (\partial_0(J\Phi))_{\Sigma} = U$$

globally define $J\Phi$ as another solution.

Then, using expression (57) or (58) it is not difficult to check that

- 1) $J^2 = -1$
- 2) J leaves ϖ invariant
- 3) $\varpi(\Phi, J\Phi) > 0$ when $\Phi \neq 0$.

4) Moreover, as Φ varies continuously with the initial data (Wald p.56) J is a continuous operator in the sense of a suitable Sobolev topology.

We have this result

Proposition 5 *Let T be an isometry of (V, g) . If T leaves invariant the Cauchy surface Σ and the source term (that is $A(Tx) = A(x)$), then the complex-structure operator J defined as above is invariant by T .*

Proof: In view of the above assumptions, not only T induces a transformation of the functions $\phi(x)$, but also a transformation of the functions U, V defined on Σ , say with a slight abuse of notation, $(TU)(y) = U(Ty)$ where $y \in \Sigma$. And T is natural with respect to restrictions to Σ , that is $(T\phi)|_\Sigma = T(\phi|_\Sigma)$. Represent ϕ by $\begin{pmatrix} U \\ V \end{pmatrix}$ and J as in (59). We obtain $[J, T] = 0$.

APPENDIX 2

The Complex Structure

Consider a real vector space \mathcal{K} endowed with a skew-symmetric form w . By complexification we obtain \mathcal{K}^C and extend w to it. It turns out that \mathcal{K}^C is endowed with a sesquilinear form

$$(\Phi; \Psi) = -iw(\Phi^*, \Psi) \quad (60)$$

for $\Phi, \Psi \in \mathcal{K}^C$. We say that Φ and Ψ are mutually *orthogonal* when $(\Phi; \Psi)$ vanishes. Let B be a linear operator acting in \mathcal{K}^C . We say that B is *symmetric* with respect to the sesquilinear form when $(B\Phi; \Psi) = (\Phi; B\Psi)$ for all Φ, Ψ . Just as well as in Hilbert spaces, symmetric operators in this sense have orthogonal eigenspaces (the proof is straightforward, but it may happen that $(\phi; \phi)$ vanishes for some $\phi \neq 0$).

A *complex-structure operator* on \mathcal{K} is a *real* linear operator J such that $J^2 = -1$. Its extension to \mathcal{K}^C is real in this sense that $(J\Psi)^* = J\Psi^*$.

Let $\Pi^\pm = \frac{1}{2}(1 \pm iJ)$. Note that Π^\pm are not real, indeed $(\Pi^+\Psi)^* = \Pi^-\Psi^*$.

"Positive-frequency" vectors can be characterized equivalently by either $\Pi^+\Psi = \Psi$ or $J\Psi = -i\Psi$.

It is easy to check that, if J is positive with respect to the skew-symmetric form, then $\Pi^+\Psi = \Psi$ implies that $(\Psi; \Psi) > 0 \quad \forall \Psi \neq 0$ (resp. $\Pi^-\Psi = \Psi$, $(\Psi; \Psi) < 0$).

Proof

Let $\Psi = M + iN$ with M and N real, then we have

$$(\Psi; \Psi) \equiv 2w(M, N) \quad (61)$$

which is real anyway. If now we assume that $\Pi^+\Psi = \Psi$, on the one hand we obtain

$$J\Psi = -i\Psi \quad (62)$$

where $-i\Psi = N - iM$. On the other hand we have $J\Psi = JM + iJN$. By identifying we obtain

$$JM = N, \quad JN = -M \quad (63)$$

Thus in (61) we can replace N by JM . Now we have

$$(\Psi; \Psi) = 2w(M, JM)$$

But positivity of J ensures that $w(M, JM)$ is positive, unless M vanishes. The case where M vanishes is necessarily that where Ψ vanishes, for $N = JM$.

In addition, invariance of w by J ensures that Π^\pm is symmetric with respect to the sesquilinear form.

Proof: Observe that (owing to $J^2 = -J$) the complex-structure operator is skew-symmetric with respect to the skew form, that is

$$w(\Phi, J\Psi) = -w(J\Phi, \Psi) \quad \forall \Phi, \Psi \quad (64)$$

iff w is invariant by J . Then, it is straightforward to check that this property entails the skew-symmetry of J with respect to the sesquilinear form, namely

$$(J\Phi; \Psi) = -iw(J\Phi^*, \Psi) = iw(\Phi^*, J\Psi) = -(\Phi; J\Psi)$$

But since $\Pi^\pm = \frac{1}{2}(1 \pm iJ)$ it follows that Π^\pm is symmetric with respect to (60).

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Replacement of D by Δ_1^\sharp Replacement of J^A by Y^A .
- [17] Strictly speaking, the notion of a reproducing kernel should be limited to operators on Hilbert spaces.